

Comment on “ Topological phase in two flavor neutrino oscillations”

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Abstract

We critically analyze the claims with regard to the relevance of topological phases in the physics of neutrino oscillation made in a recent paper [Phys. Rev. D 79, 096013 (2009)] and point out some inappropriate exaggerations and misleading statements. We find that the π phase described in this paper, while interesting, is an artefact of two major approximations made in the paper. We point out a more robust and more familiar π phase in the neutrino oscillation formulae which can be interpreted as a pure Pancharatnam phase. We also make some relevant remarks on the distinction between the geometric and the topological phase made in the commented paper.

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In a recent paper [1] it has been claimed that “ for the minimal case of two flavors and CP conservation, there is a geometric interpretation of the neutrino oscillation formulae for the survival and detection probabilities of neutrino species”. In this paper we first recall the derivation of the main result in [1] with a slightly different notation and then show that (i) there is a more robust and more familiar π phase in neutron oscillation formulae than the one discussed in [1] which can be seen as a pure Pancharatnam phase, (ii) the geometric interpretation and the π phase discussed in [1] is an artefact of two important approximations made in the paper and thus lacks fundamental significance, (iii) the geometric interpretation belongs only to an aspect of the neutron oscillation formulae and not to the formulae themselves, (iv) the distinction between the topological phase and geometric phase introduced in the paper is inappropriate and (v) discuss certain misleading statements made in the paper.

As in [1], let $|\nu_\alpha\rangle$ and $|\nu_\beta\rangle$ be the two flavor eigenstates represented by the two antipodal points lying along the z-axis and let $|\theta, \pm\rangle$ be the two orthogonal mass eigenstates which lie on the line making an angle θ with respect to the z-axis. Let the initial state be

$$|\nu_\alpha\rangle = c_+|\theta_1, +\rangle + c_-|\theta_1, -\rangle, \quad (1)$$

where the coefficients $\nu_{\alpha+}$ and $\nu_{\alpha-}$ of [1] have been replaced by c_+ and c_- , the rest of the being the same. This state evolves in time t to the state

$$|\nu_\alpha\rangle' = e^{iD_+}c_+|\theta_2, +\rangle + e^{iD_-}c_-|\theta_2, -\rangle, \quad (2)$$

where D_+ and D_- are the dynamical phases acquired during the adiabatic evolution of the states $|\theta, \pm\rangle$ from $|\theta_1, \pm\rangle$ to $|\theta_2, \pm\rangle$.

The survival probability P_α and the transition probability P_β are given by,

$$P_\alpha = |\langle \nu_\alpha | \nu_\alpha \rangle'|^2 = |c_+|^2 |\langle \nu_\alpha | \theta_2, + \rangle|^2 + |c_-|^2 |\langle \nu_\alpha | \theta_2, - \rangle|^2 + (c_+^* c_- e^{i(-D_- + D_+)}) \langle \nu_\alpha | \theta_2, + \rangle^* \langle \nu_\alpha | \theta_2, - \rangle + c.c.), \quad (3)$$

$$P_\beta = |\langle \nu_\beta | \nu_\alpha \rangle'|^2 = |c_+|^2 |\langle \nu_\beta | \theta_2, + \rangle|^2 + |c_-|^2 |\langle \nu_\beta | \theta_2, - \rangle|^2 + (c_+^* c_- e^{i(-D_- + D_+)}) \langle \nu_\beta | \theta_2, + \rangle^* \langle \nu_\beta | \theta_2, - \rangle + c.c.). \quad (4)$$

We first note a rigorous result within the two-flavor model which does not depend on any approximations. Since it has been assumed that there is no decay, P_α and P_β must add to 1. An inspection of eqns. (3) and (4) makes it obvious that this is possible only if the cross terms in the two equations add to zero. This implies that the complex numbers $A_\alpha = \langle \nu_\alpha | \theta_2, + \rangle^* \langle \nu_\alpha | \theta_2, - \rangle$ and $A_\beta = \langle \nu_\beta | \theta_2, + \rangle^* \langle \nu_\beta | \theta_2, - \rangle$ should be equal in magnitude and differ in phase by π , i.e. the phase of the number $A_\beta^* A_\alpha$ must be equal to $\pm\pi$. Now

$$A_\beta^* A_\alpha = \langle \nu_\beta | \theta_2, + \rangle \langle \nu_\beta | \theta_2, - \rangle^* \langle \nu_\alpha | \theta_2, + \rangle^* \langle \nu_\alpha | \theta_2, - \rangle. \quad (5)$$

Using the fact that $\langle \nu_\beta | \theta_2, - \rangle^* = \langle \theta_2, - | \nu_\beta \rangle$ etc. and rearranging the terms we get

$$A_\beta^* A_\alpha = \langle \nu_\beta | \theta_2, + \rangle \langle \theta_2, + | \nu_\alpha \rangle \langle \nu_\alpha | \theta_2, - \rangle \langle \theta_2, - | \nu_\beta \rangle. \quad (6)$$

By Pancharatnam's theorem, the phase of the complex number on the right hand side of Eq.(6) is equal to half the solid angle subtended by the closed geodesic curve starting at the state $|\nu_\beta\rangle$, passing through the states $|\theta_2, -\rangle$, $|\nu_\alpha\rangle$, $|\theta_2, +\rangle$ and ending at $|\nu_\beta\rangle$, which is a great circle on the Poincaré sphere. This phase is obviously equal in magnitude to π . This is the well known π phase between the oscillations of intensities of the two different flavours which is seen here as a pure Pancharatnam phase of magnitude π . It follows from the condition of unitarity and is an elegant example of internal consistency of different principles of physics having apparently different origins. Within the two-state model, this phase is independent of any approximations. It can be shown easily that this π phase does not depend even on the adiabatic approximation. Let us also note that (a) this phase is independent of the phases of the individual states occurring in Eq.(6) and (b) this phase can be looked upon as the phase acquired by the state $|\nu_\alpha\rangle$ if it evolved along the closed great circle under the action of a constant unitary hamiltonian that represents rotation about an axis perpendicular to the great circle, i.e. under an SU(2) element that represents a 2π rotation on the Poincaré sphere about this axis.

If one stares at Eqs.(3) and (4) for a while it becomes obvious that the content of the above result can be exactly simulated by the following polarization experiment: Let polarization states $|\theta_2, +\rangle$ and $|\theta_2, -\rangle$ be incident

on the two slits of an interferometer and a polarizer that passes the state $|\nu_\alpha\rangle$ be placed in front of the screen and the position of the fringes noted. Let the polarizer now be replaced by one that passes the state $|\nu_\beta\rangle$ and the shift in the fringes measured. The above result says that irrespective of the incident states the measured phase shift must be equal to π in magnitude.

The above result has in fact been demonstrated in optical interference experiments [2] using the two-state system of light polarization. The results shown in Fig. 3 of [2] show that a rotation of a linear polarizer through 90° always results in a phase shift of $\pm\pi$ irrespective of the polarization states of the interfering beams.

We next come to the π phase discussed by Mehta [1]. In the cross term on the right hand side of Eq. (4), if we substitute for c_+ and c_- from Eq.(1) we get, after rearranging the terms, the product $\langle \nu_\alpha | \theta_1, + \rangle \langle \theta_2, + | \nu_\beta \rangle \langle \nu_\beta | \theta_2, - \rangle \langle \theta_1, - | \nu_\alpha \rangle$ multiplying the exponential term. To make this product correspond to evolution of a state along a closed great circle one needs two more terms $\langle \theta_2, - | \theta_1, - \rangle$ and $\langle \theta_1, + | \theta_2, + \rangle$ which are missing from the product. To compensate for the missing terms, the author first sacrifices the arbitrariness of the phases of the individual states in the product and then makes the approximation that the hamiltonian is CP non-violating. Let us note that the approximation of adiabatic evolution of the states $|\theta_1, - \rangle$ and $|\theta_1, + \rangle$ to the states $|\theta_2, - \rangle$ and $|\theta_2, + \rangle$ has already been made. Under these two approximations, the author argues rightly that the phases of the missing terms are accounted for exactly if the phases of the pairs of states $|\theta_1, - \rangle$, $|\theta_2, - \rangle$ and $|\theta_1, + \rangle$, $|\theta_2, + \rangle$ in the product are related by parallel transport and that the phase of the product is then equivalent to that acquired in a unitary evolution along a great circle under a constant hamiltonian, i.e. equal in magnitude to π . If any of the two approximations is dropped the result is no more true. In fact the author shows in a separate paper [3] that if the adiabatic approximation is retained but the hamiltonian is allowed to be CP-violating, the product is no longer equivalent to evolution along a closed great circle and the phase of the product is no longer π but is equal to that determined by the solid angle of the distorted curve! The phase of magnitude π is thus an artefact of the restriction on the hamiltonian.

In order to evaluate the significance of the result let us first consider the case of evolution in free space or in constant density matter where the states $|\theta_1\rangle$ and $|\theta_2\rangle$ are the same. In this case the variations of the flavor intensities are pure sinusoidal oscillations. The sinusoidal oscillation has three attributes: (A) amplitude of the oscillation which is determined by the mod-

ulus of the cross terms in Eqs. (3) and (4), (B) frequency of the oscillation which is determined by the dynamical phase term containing D_+ and D_- and (C) a constant phase which is equal to 0 in case of the survival probability given by Eq. (3) and equal in magnitude to π in case of the transition probability given by Eq. (4). The main results of the paper are concerned with only the attribute (C), i.e. the constant phase of the oscillation. Someone who had never heard of the Pancharatnam phase would fix this constant phase trivially by requiring that the survival probability P_α and the transition probability P_β be equal to 1 and 0 respectively at time $t=0$. Surely it can be seen as a Pancharatnam phase, but the claim on this basis, as in the abstract of the paper, that “the neutron oscillation formulae have a geometric interpretation” is, in our view, a gross exaggeration. The statement “More precisely, the standard result for neutrino oscillations is in fact a realization of the Pancharatnam topological phase” on page 9 is also an inappropriate exaggeration. It is equivalent to claiming in the first example discussed in this comment that unitarity is a realization of the Pancharatnam phase !. In our view the frequency of the oscillation determined by the dynamical phase and the amplitude of the oscillation determined by the mixing angle are at least as important parts of the neutron oscillation formulae as the absolute phase of the oscillation.

When variable matter density is introduced, the dynamical phase is no more a linearly varying function of time nor is the amplitude of the cross term constant in time. Both these effects are just as important to the variation of flavour intensities along the path as the geometric part of the phase and are parts of the neutron oscillation formulae. In fact in the adiabatic limit, since the dynamical phase is large compared to the geometric phase by definition, the variation of dynamical phase due to the presence of variable matter density could easily dominate over the geometric term. Moreover, as discussed above, the value π for the phase found by the author is a consequence of the restriction on the hamiltonian and the adiabatic approximation and is , therefore, not fundamental. It is just a special value obtained under specified constraints.

A distinction in terminology between the π phase obtained when CP violation is absent and the non- π phase obtained when it is present has been made in the paper, the former being called topological and the latter geometric. We find this distinction unnatural and inappropriate since both are parts of a single phenomenon and are manifestations of the same basic singularity associated with the SU(2) group. A unified description of this

singularity has been described in several papers. In the context of adiabatic evolution, a three-dimensional generalization of the sign change rule has been described in [4]. In the context of nonadiabatic evolution an operational description of this singularity has been given in [5, 6, 7] and experimental demonstrations with polarization of light have been reported in [6, 2, 8]. We briefly recall this work below.

Consider the evolution of a spin-1/2 state under the action of a hamiltonian which is a function of three parameters x , y and z . First let us consider evolution along a closed circuit in a two-dimensional space, i.e. the plane $y = 0$. Let the hamiltonian be degenerate at an isolated point $(x_0, 0, z_0)$ in this plane. Let the parameters of the hamiltonian be changed so that the closed circuit moves from a condition where it does not encircle the point $(x_0, 0, z_0)$ to a condition where it does. Then the considerations in [1] say that at the transition point where the boundary of the closed circuit crosses the degeneracy, there is a sudden jump of magnitude π in the phase of the state, assuming that the dynamical phase has been subtracted out. We point out that at the point of crossing of the degeneracy the adiabatic approximation must break down. It was pointed out in [4] that if the same circuit were located in the plane $y = \epsilon$, where ϵ is small and the same motion of the circuit carried out so that this time there is no actual crossing of the degeneracy, there is a measurable phase shift of a magnitude approximately equal to, $+\pi$ whereas if the same operation were carried out in the plane $y = -\epsilon$ there is a measurable phase shift of a magnitude approximately equal to $-\pi$ (only the relative sign being important) [9]. It was further shown that if the circuit were taken around a closed loop such that it loops the degeneracy, there is a measurable phase shift equal to $\pm 2\pi$. A monopole of strength 1/2 located at the degeneracy point gives a good unified description of the two effects which have been termed differently in [1].

To consider the more general nonadiabatic case, an element of $SU(2)$ which corresponds to a 2π rotation about any axis on the Poincaré sphere is represented by the matrix $-\mathbf{1}$. Any state acted upon by this element therefore acquires a phase of magnitude π . This is the well-known phenomenon of “ 4π spinor symmetry”. States with different polar angles with respect to the rotation axis execute small circles of different diameter but the total phase acquired in one full cycle is always of magnitude π . There is, however, another dimension to the problem. A phase shift has a magnitude as well as a sign. If the phase of a state evolving under the above hamiltonian were continuously monitored with an interferometer with reference to some reference state $|R\rangle$,

the total measured phase shift will be $+\pi$ for some states and $-\pi$ for others. At critical values of the parameters, the phase shift jumps suddenly from being $+\pi$ to being $-\pi$. This has been verified in polarization experiments [6, 2]. This is due to the presence of phase singularities which can be identified as follows. Let $|R\rangle$ stand for the reference state in the interferometer and $|\tilde{R}\rangle$ the diametrically opposite state on the sphere which is orthogonal to $|R\rangle$. At points during the evolution where the evolving state is equal to $|\tilde{R}\rangle$ the interference pattern vanishes and the phase is undetermined. In the vicinity of this point the phase shifts vary sharply. In general, if a state $|I\rangle$ incident on one arm of an interferometer undergoes an $SU(2)$ transformation U which is a function of some variable parameters to yield a final state $|F\rangle$ which interferes with a state $|R\rangle$ in the reference arm then a closed cycle of the parameters of U around an isolated point at which $|F\rangle = |\tilde{R}\rangle$ yields a total phase shift equal to $\pm 2n\pi$ where n is an integer index representing the strength of the singularity. Some examples of such phase shifts have been demonstrated in interference experiments [8]. This result is conceptually simpler than the adiabatic result as it does not depend on subtraction of a large dynamical phase from the total phase. Note that in both the above discussions the phase shift associated with the singularity is of magnitude $2n\pi$ and not π .

Finally we point out that at the end of page 2 of [1], the statement “Now this open loop (noncyclic) Schrodinger evolution of a quantum state over a time τ can be closed by a collapse of the time-evolved quantum state at τ onto the original state at $\tau = 0$ by the shortest geodesic curve joining the two states in the ray space [10].” is a misrepresentation of historical facts. A similar statement made by Samuel on page 960 of [11] is also false. What was said in Samuel and Bhandari [10] was just the opposite. It was stated repeatedly on page 2341 of [10] that the final state can be connected to the initial state by *any* geodesic arc. After a careful discussion with the first author of [10] the author of [1] has stated in [12] that the word “any” in “any geodesic arc” refers to any of the gauge copies of the geodesic in the N - space. Our response to this is that the expression “any geodesic arc” includes all geodesic arcs in N , i.e. those that project down to the shorter geodesic as well as those that project down to the longer geodesic in the ray space. The footnote from [10] cited in [12] is merely a description of a property of the shorter geodesic and does not constitute a restriction on the definition of the noncyclic geometric phase. In fact the footnote does not form part of the discussion of the noncyclic geometric phase on p. 2341 where the expression

“any geodesic curve” occurs again. In response to the following comment made in [12] : “the author has been caught in the unfortunate position of having been scooped by himself”, a statement we do not understand, we reiterate that the first clear statement that the noncyclic geometric phase should be defined as half the solid angle of the area obtained by closing the open curve in the ray space with the shortest geodesic arc connecting the final state to the initial state was made in [5], i.e. [R. Bhandari, Phys. Lett. **A 157**, 221 (1991)] and not in [10], i.e. [J. Samuel and R. Bhandari, Phys. Rev. Lett. **60**, 2339 (1988)]. This led to the prediction of observable $\pm\pi$ phase jumps in two-state systems which were later verified in interference experiments [6, 2] and to the nonmodular view of the topological phase, thus constituting a conceptual advance in the subject. The restriction to the shorter geodesic is thus much more than a footnote. It may also be pointed out that the definition of the noncyclic geometric phase proposed in [5] as the difference of the total Pancharatnam phase of the evolving state and the dynamical phase as defined by Aharonov and Anandan does not depend on a geodesic rule and is thus particularly useful for systems with more than two states where the geometry of the ray space can not be easily visualized. An extension of the definition to the case of an arbitrary reference state has been proposed in [7]. To end this discussion we note that the fact that the shortest geodesic rule was not used in [10] was also pointed out in [4].

To sum up, the net content of [1] would be precisely summarized if the abstract of the paper read: “We show that, under the adiabatic approximation, the phase appearing in the neutrino oscillation formulae has a geometric contribution which, under the constraint of CP non-violation, is equal in magnitude to π ”. Considering that the phase in a quantum evolution in general has a geometric part, this is not very significant.

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